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Abstract

Wireless networks have shown a rapid growth over the past two decades and now play a key role in new generation telecommunications networks. The physical medium of wireless networks is the radio spectrum, a scarce resource which is becoming extremely congested and needs to be allocated in more effective ways. Since the early 1980s several optimization models have been developed to design wireless networks.

In this paper we propose a pure 0-1 formulation which is able to model a very general situation in which both emission powers and operating frequencies can be optimized. In contrast with the classical mixed integer formulation, where powers are represented by continuous variables, we consider only a finite set of transmitting powers. The ensuing model has two major advantages: it better fits the usual practice and minimizes the numerical problems produced by the interaction of continuous and 0-1 decision variables. A crucial ingredient of our approach is an effective basic formulation for the single knapsack problem representing the coverage condition of a receiver. This formulation is based on the well-known lifted GUB cover inequalities introduced by Wolsey and its core is a slight extension of the exact formulation proposed by Wolsey for the GUB knapsack polytope with two GUB constraints. In the specific framework of our real-life problem the two GUB constraints case corresponds to the very common situation in which only one major interferer is present. The effectiveness of such formulation is assessed by comprehensive computational results.

Keywords: Wireless Network Design, 0-1 Linear Programming, GUB cover inequalities.

1 Introduction

Wireless communication systems constitute one of the most pervasive phenomena of everyday life. Television and radio programs are distributed through broadcasting networks (both terrestrial and satellite), mobile communication is ensured by cellular networks, internet is provided through broadband access networks. Moreover, a number of security services is provided by ad-hoc wireless networks. All these networks have grown very rapidly during the last decades, generating dramatic congestion of radio resources. Wireless networks provide different services and rely on different technologies and standards. Still, they share a common feature: they all need to reach users scattered over a target area with a radio signal that must be strong enough to prevail against other unwanted signals. The perceived quality of service thus depends on several signals, wanted and unwanted, generated from a large number of transmitting devices. Due to the increasing size of the new generation networks, co-existing in an extremely congested radio spectrum and subject to local and international constraints, establishing suitable emission powers for all the transmitters has become a very difficult task, which calls for sophisticated optimization techniques.

Since the early 1980s several optimization models have been developed to design wireless networks, that is to localize and configure the transmitters by assigning operating frequencies and emission powers.

In the scientific literature, the emission powers have been mostly represented as continuous decision variables. This choice typically yields ill-conditioned constraint matrices and requires the introduction of very large coefficients to model disjunctive constraints. The corresponding relaxations are very weak and state-of-the-art Mixed-Integer Linear Programming solvers are often affected by numerical instability. The use of continuous decision variables also contrasts with the telecommunications practice. In fact, the actual design specifications of real life antennas are always expressed as rationals with bounded precision and, consequently, assume a finite number of values.

Motivated by the above remarks we propose a pure 0-1 formulation for the problem that is obtained by considering only a finite set of power values. This formulation has two basic advantages: first, the ensuing model better fits the usual practice and, second, the numerical problems produced by the continuous variables are sensibly reduced. Indeed, the new approach allows us to find better solutions to large practical instances with less computational effort. Also, since the feasible powers are well spaced over the power spectrum, the optimal solutions are not affected by limited but quite common deviations from the real parameters. In addition, the model fits the common network planning practice of considering a small number of power values and it directly models power restrictions that are often imposed by the technology (see, e.g., [17]). It is not rare the situation where only two power values are allowed, i.e. on/off ([26]). Finally, the new approach easily allows for generalizations of the model, such as power consumption minimization or antenna diagram optimization.

For our purposes, a wireless network can be described as a set of transmitters B distributing a telecommunication service to a set of receivers T. A receiver is said to be *covered* (or *served*) by the network if it receives the service within a minimum level of quality. Transmitters and receivers are characterized by a number of location and radio-electrical parameters (e.g. geographical coordinates, emission power, transmission frequency). The *Wireless Network Design* problem (WND) consists of establishing suitable values for such parameters with the goal of maximizing the coverage (or a revenue associated with the coverage).

Each transmitter $b \in B$ emits a radio signal with power $p_b \in [0, P_{\max}]$. The power $p(t)$ received in t from transmitter b is proportional to the emitted power p_b by a factor $\tilde{a}_{tb} \in [0, 1]$, i.e. $p(t) = \tilde{a}_{tb} \cdot p_b$. The factor $\tilde{a}_{tb}$ is called *fading coefficient* and summarizes the reduction in power that a signal experiences while propagating from b to t. The value of a fading coefficient depends on many factors (e.g. environment (urban or rural), distance between the communicating devices, presence of obstacles, antenna patterns) and is commonly computed through a suitable propagation model. For a detailed discussion on all technical issues we refer the reader to [25].

To simplify the discussion, we assume here that all the transmitters of the network operate at the same frequency. This assumption is dropped in Section 5 where we describe the real-life application which motivated our developments. Among the signals received from transmitters in B, receiver t can select a *reference signal* (or *server*), which is the one carrying the service. All the other signals are interfering.

A receiver t is regarded as served by the network, specifically by server

$\beta \in B$, if the ratio of the serving power to the sum of the interfering powers (*signal-to-interference ratio* or *SIR*) is above a threshold δ' [25]:

$$\frac{\tilde{a}_{t\beta} \cdot p_\beta}{N + \sum_{b \in B(t)\setminus\{\beta\}} \tilde{a}_{tb} \cdot p_b} \geq \delta' \tag{1}$$

Note the presence of the system noise $N > 0$ among the interfering signals. Moreover, we introduce a set $B(t) \subseteq B$ to denote the subset of transmitters whose signals can be successfully detected by t. Such subset includes the transmitters $b \in B$ that satisfy the SIR expression (1) when they emit at maximum power and only noise is present, i.e. $B(t) = \{b \in B : \frac{\tilde{a}_{tb} \cdot P_{\max}}{N} \geq \delta'\}$. Since each transmitter in $B(t)$ is associated with a unique received signal, in what follows we will also refer to $B(t)$ as the set of signals received by t.

The SIR threshold δ' is a parameter whose value depends on the technology and on the wanted quality of service. By letting $\delta = -N \cdot \delta' < 0$ and letting:

$$a_{tb} \quad = \quad \begin{cases} \tilde{a}_{tb} & \text{if } b = \beta \\ \delta' \cdot \tilde{a}_{tb} & \text{otherwise} \end{cases}$$

for every $b \in B(t)$, then the inequality (1) can be transformed into the following linear inequality (*SIR inequality*):

$$\sum_{b \in B(t)\setminus\{\beta\}} a_{tb} \cdot p_b - a_{t\beta} \cdot p_\beta \leq \delta \tag{2}$$

For every $t \in T$, we have one inequality of type (2) for each potential server $\beta \in B(t)$. Receiver t is served if at least one such inequality is satisfied or, equivalently, if the following disjunctive constraint is satisfied:

$$\bigvee_{\beta \in B(t)} \left(\sum_{b \in B(t)\setminus\{\beta\}} a_{tb} \cdot p_b - a_{t\beta} \cdot p_\beta \leq \delta \right) \tag{3}$$

The above disjunction can be represented by a family of linear constraints in the p variables by introducing, for each $t \in T$ and each $b \in B(t)$, a binary variable x_{tb} that is equal to 1 if t is served by b and to 0 otherwise. For each

$\beta \in B(t)$, the following constraint is then introduced:

$$\sum_{b \in B(t) \setminus \{\beta\}} a_{tb} \cdot p_b - a_{t\beta} \cdot p_\beta - M \cdot (1 - x_{t\beta}) \leq \delta \tag{4}$$

where M is a large positive constant (big-M). Indeed, when $x_{t\beta} = 1$ then (4) reduces to (2); when instead $x_{t\beta} = 0$ and M is sufficiently large, (4) is satisfied for any feasible power vector and becomes redundant. Constraints of type (4) appear in the mixed-integer linear programs (MILPs) for the WND presented in several papers in different application contexts, such as radio and video broadcasting (e.g. [19, 20]), GSM (e.g. [21]), UMTS (e.g. [2, 11, 16, 22]), WiMAX [8]. Such linear programs are informally called *big-M formulations*.

WND instances of some practical interest typically correspond to very large MILPs. In principle, such programs can be solved by standard Branch-and-Cut and by means of effective commercial solvers such as ILOG Cplex [6]. However, it is well-known that the presence of a great number of constraints of type (4) results in ill-conditioned instances, due to the large variability of the fading coefficients, and weak bounds, due to the presence of the big-M coefficients. Furthermore, the resulting coverage plans are often unreliable (e.g. [16, 19]). In practice, only small-sized instances can actually be solved to optimality.

Former attempts to directly address these issues include different reformulations of the WND by applying classical decomposition approaches, such as Dantzig-Wolfe [18] and Benders' decomposition [22, 24]. We follow here a different path, as we will show in the next section: namely, we discretize the continuous variables and consider only a finite number of feasible values. We stress that discretization is a classical tool in combinatorial optimization (e.g. [10]) and in telecommunication modeling (e.g. [4, 12, 17]) but, to our best knowledge, no effort has been made to go beyond the simple use of discretized SIR inequalities and replace them by more combinatorial inequalities. This is the main goal of this paper.

2 A Power-Indexed formulation for the WND

As we said in the previous section, a classical and much exploited model for the WND belongs to the class of the so-called big-M formulations and writes as:

$$
\begin{aligned}
\max \quad & \sum_{t\in T}\sum_{b\in B(t)} r_t \cdot x_{tb} && && (BM)\\
\text{s.t.} \quad & \sum_{b\in B(t)\setminus\{\beta\}} a_{tb}\cdot p_b - a_{t\beta}\cdot p_\beta - M\cdot(1-x_{t\beta}) \le \delta && t\in T, \beta\in B(t) && (5)\\
& \sum_{b\in B(t)} x_{tb} \le 1 && t\in T && (6)\\
& 0\le p_b \le P_{max} && b\in B &&\\
& x_{tb}\in\{0,1\} && t\in T, b\in B(t) &&
\end{aligned}
$$

where r_t is the revenue (e.g. population, number of customers, expected traffic demand) associated with receiver $t \in T$ and the objective function is to maximize the total revenue. Constraint (5) is the SIR inequality (4) introduced in Section 1 and constraint (6) ensures that each receiver is served at most once.

Technology-dependent versions of (BM) can be obtained from the basic formulation by including suitable constraints or even new variables. For example, in the case of WiMAX networks, a knapsack constraint involving the service variables x_{tb} is added to (BM) to model the bandwidth capacity of each transmitter $b \in B$ (see Section 5). In the case of antenna diagram design, the number of power variables associated with each transmitter b is multiplied by 36 to represent the power emissions along the 36 directions which approximate the horizontal radiation pattern, and new constraints are included to represent physical relations between different directions [19].

As observed in the introduction, the problem (BM) has serious drawbacks both in terms of dimension of the solvable instances and of numerical instability. A way to tackle these issues is that of restricting the variables p_b to assume value in the finite set $\mathcal{P} = \{P_1, \ldots, P_{|\mathcal{P}|}\}$ of feasible power values, with $P_1 = 0$ (*switched-off value*), $P_{|\mathcal{P}|} = P_{max}$ and $P_i > P_{i-1}$, for $i = 2, \ldots, |\mathcal{P}|$. To this end, we introduce a binary variable z_{bl}, which is 1 iff b emits at power P_l. Since b is either switched-off or emitting at a positive value in $\mathcal{P}$, we have:

$$\sum_{l\in L} z_{bl} = 1 \qquad b\in B$$

where $L = \{1, \ldots, |\mathcal{P}|\}$ is the set of power value indices or simply *power* *levels*. Then we can write:

$$p_b = \sum_{l \in L} P_l \cdot z_{bl} \qquad b \in B \tag{7}$$

By substituting (7) in (5), we obtain the following SIR constraint that only involves 0-1 variables:

$$\sum_{b \in B(t) \setminus \{\beta\}} a_{tb} \sum_{l \in L} P_l \cdot z_{bl} \; - a_{t\beta} \sum_{l \in L} P_l \cdot z_{\beta l} \; - M \cdot (1 - x_{t\beta}) \leq \delta$$

The following *discrete big-M formulation* (DM) for the WND with a finite number of power values directly derives from (BM):

$$\begin{array}{llll}
\max & \displaystyle\sum_{t \in T} \sum_{b \in B(t)} r_t \cdot x_{tb} & & (DM) \\
\text{s.t.} & \displaystyle\sum_{b \in B(t) \setminus \{\beta\}} a_{tb} \sum_{l \in L} P_l \cdot z_{bl} - a_{t\beta} \sum_{l \in L} P_l \cdot z_{\beta l} + M \cdot x_{t\beta} \leq \delta + M & t \in T, \beta \in B(t) & (8) \\
& \displaystyle\sum_{b \in B(t)} x_{tb} \leq 1 & t \in T & \\
& \displaystyle\sum_{l \in L} z_{bl} = 1 & b \in B & (9) \\
& x_{tb} \in \{0, 1\} & t \in T, b \in B(t) & \\
& z_{bl} \in \{0, 1\} & b \in B, l \in L &
\end{array}$$

Note that, thanks to (7), every p_b also satisfies $0 \leq p_b \leq P_{max}$. As a consequence, the box constraints on p_b and thus variable p_b is dropped from the formulation.

The Power-Indexed formulation is obtained from (DM) by substituting each *knapsack SIR constraint* (8) with a suitable set of inequalities called *lifted GUB cover inequalities (LGUB)* [29].

Before showing this, we recall some related definitions and concepts introduced in [29]. We consider the set of binary points $Y = P \cap B^n$, where

$P \subseteq R^n_+$ is the polytope defined by:

$$\sum_{j \in N_1} a_j \cdot y_j - \sum_{j \in N_2} a_j \cdot y_j \leq a_0 \tag{10}$$

$$\sum_{j \in S_i} y_j \leq 1 \qquad \text{for } i \in I_1 \cup I_2$$

$$y \in R^n_+,$$

where $N = N_1 \cup N_2$, $N_1 \cap N_2 = \emptyset$, $a_j > 0$ for $j \in N$, $\bigcup_{i \in I_1} S_i = N_1$, $\bigcup_{i \in I_2} S_i = N_2$ and, finally $S_i \cap S_l = \emptyset$ if $i, l \in I_k$ with $i \neq l$ for $k = 1, 2$. In other words, the variables of the knapsack are partitioned into a number of subsets, and at most one variable can be set to 1 for each subset. In addition, each subset is entirely contained either in N_1 or N_2.

A set $C = C_1 \cup C_2$ is a *GUB cover* for Y if:

(i) $C_k \subseteq N_k \qquad$ for $k = 1, 2$

(ii) $|C_k \cap S_i| \leq 1 \quad$ for $i \in I_k$ and $k = 1, 2$

(iii) $\sum_{j \in C_1} a_j - \sum_{j \in C_2} a_j > a_0$

With the GUB cover C we associate the following sets:

$$I_k^+ = \{i \in I_k : C_k \cap S_i \neq \emptyset\} \qquad \text{for } k = 1, 2$$

$$S_i^+ = \{j \in S_i : a_j \geq a_l \text{ for } l \in C_1 \cap S_i\} \quad \text{for } i \in I_1^+$$

$$S_i^+ = \{j \in S_i : a_j \leq a_l \text{ for } l \in C_2 \cap S_i\} \quad \text{for } i \in I_2^+$$

In [29], Wolsey proves that if $C = C_1 \cup C_2$ is a *GUB cover*, the following inequality is valid for Y:

$$\sum_{i \in I_1^+} \sum_{j \in S_i^+} y_j \leq |C_1| - 1 + \sum_{i \in I_2^+} \sum_{j \notin S_i^+} y_j + \sum_{i \in I_2 \setminus I_2^+} \sum_{j \in S_i} y_j$$

When $I_2^+ = I_2$ and $|I_2| = 1$, such valid inequality reduces to:

$$\sum_{i \in I_1^+} \sum_{j \in S_i^+} y_j + \sum_{i \in I_2^+} \sum_{j \in S_i^+} y_j \leq |C_1| \tag{11}$$

Now, let us focus on a single knapsack constraint (8) of (DM) associated with testpoint $t \in T$ and server $\beta \in B(t)$, along with constraints (9) for $b \in B(t)$ and the valid inequality $x_{t\beta} \leq 1$. We can cast this into the GUB framework introduced by Wolsey by making the following associations.

$$N_1 = \{(b,l) : \ b \in B(t) \backslash \{\beta\}, \ l \in L\} \ \cup \ \{(t,\beta)\}$$
$$N_2 = \{(\beta,l) : \ l \in L\}$$

Observe that, with a slight abuse of notation, in the definition of N_1 we are also including index (t,β) corresponding to variable $x_{t\beta}$. Similarly, we let:

$$I_1 = \{b : \ b \in B(t) \backslash \{\beta\}\} \ \cup \ \{(t,\beta)\}$$
$$I_2 = \{\beta\}$$

Indeed, for each $b \in B(t)$ at most one variable z_{bl} can be equal 1, for $l \in L$, and we have $S_b = \{(b,l) : l \in L\}$ for all $b \in B(t)$. Also, we let $S_{t,\beta} = \{(t,\beta)\}$ be the singleton corresponding to variable $x_{t\beta}$. Observe that we have $N_1 = S_{t,\beta} \cup (\bigcup_{b \in B(t) \backslash \{\beta\}} S_b)$ and $N_2 = S_\beta$.

We now translate conditions (i), (ii) and (iii) into our setting. To this purpose, consider first the coverage condition (2) corresponding to receiver $t \in T$ with server $\beta \in B(t)$. Suppose that the server β is emitting at power value $p_\beta = P_\lambda$, for some $\lambda \in L$. Let $\Gamma = \{b_1, \ldots, b_{|\Gamma|}\} \subseteq B(t) \backslash \{\beta\}$ be a set of interferers (for t when β is its server) and let $q_1, \ldots, q_{|\Gamma|}$ be power levels for each interferer in Γ such that:

$$a_{tb_1} \cdot P_{q_1} + \cdots + a_{tb_{|\Gamma|}} \cdot P_{q_{|\Gamma|}} - a_{t\beta} \cdot P_\lambda > \delta \tag{12}$$

In other words, receiver t is not served when t is assigned to server β emitting at power value P_λ, and the interferers $b_1, \ldots, b_{|\Gamma|}$ are emitting at power values $p_{b_1} = P_{q_1}, \ \ldots, \ p_{b_{|\Gamma|}} = P_{q_{|\Gamma|}}$, respectively.

By letting $C_1 = \{(b_i, q_i) : i = 1, \ldots, |\Gamma|\} \cup \{(t,\beta)\}$ and $C_2 = \{(\beta, \lambda)\}$, it follows that $C = C_1 \cup C_2$ is a cover of (8). Also, it is not difficult to see that

C is a GUB cover, since $C_1 \subseteq N_1$, $C_2 \subseteq N_2$, $|C_1 \cap S_b| \leq 1$, for all $b \in I_1$ and $|C_2 \cap S_\beta| = 1$. We also have $I_1^+ = \Gamma \cup \{(t,\beta)\}$ and $I_2^+ = \{\beta\}$.

Since $a_{tb} \cdot P_l < a_{tb} \cdot P_{l+1}$ for all $b \in B(t)$ and $l = 1, \ldots, |L| - 1$, we have that $S_b^+ = \{(b, q_i), (b, q_{i+1}), \ldots, (b, q_{|L|})\}$ for $b \in B(t)\backslash\{\beta\}$, $S_{t,\beta}^+ = \{(t,\beta)\}$ and $S_\beta^+ = \{(\beta, 1), \ldots, (\beta, \lambda)\}$.

It follows from (11) that, for $t \in T$, $\beta \in B(t)$, the inequality

$$x_{t\beta} + \sum_{l=1}^{\lambda} z_{\beta l} + \sum_{i=1}^{|\Gamma|} \sum_{j=q_i}^{|L|} z_{b_i j} \leq |\Gamma| + 1 \tag{13}$$

is valid for the set of binary vectors satisfying (8) and (9).

Now, for all the subsets of interferers $\Gamma \subseteq B(t)\backslash\{\beta\}$, denote by $L^I(t, \beta, \lambda, \Gamma)$ the set of $|\Gamma|$-tuples $q \in L^{|\Gamma|}$ satisfying (12). The following proposition follows immediately by the validity of (13):

Proposition 1 *Given $t \in T$, $\beta \in B(t)$, the family of inequalities:*

$$x_{t\beta} + \sum_{l=1}^{\lambda} z_{\beta l} + \sum_{i=1}^{|\Gamma|} \sum_{j=q_i}^{|L|} z_{b_i j} \leq |\Gamma| + 1 \tag{14}$$

defined for $\Gamma \subseteq B(t)\backslash\{\beta\}$, $\lambda \in L$, $q \in L^I(t, \beta, \lambda, \Gamma)$, is satisfied by all the binary solutions of (8) and (9).

One can show that the reverse is also true, namely all binary solutions to (14) and (9) also satisfy (8) (for the proof we refer the reader to [7]). It follows that the following formulation, that we call Power-Indexed (PI), is valid for the WND (with finite set of power values):

$$
\begin{aligned}
\max \quad & \sum_{t\in T}\sum_{b\in B(t)} r_t \cdot x_{tb} && && (PI)\\
\text{s.t.} \quad & x_{t\beta} + \sum_{l=1}^{\lambda} z_{\beta l} + \sum_{i=1}^{|\Gamma|}\sum_{j=q_i}^{|L|} z_{b_i j} \le |\Gamma| + 1 && t\in T,\ \beta\in B(t),\ \Gamma\subseteq B(t)\backslash\{\beta\}, && \\
& && \lambda\in L,\ q\in L^I(t,\beta,\lambda,\Gamma) && (15)\\
& \sum_{b\in B(t)} x_{tb} \le 1 && t\in T && (16)\\
& \sum_{l\in L} z_{bl} = 1 && b\in B && (17)\\
& x_{tb}\in\{0,1\} && t\in T, b\in B(t) && (18)\\
& z_{bl}\in\{0,1\} && b\in B, l\in L && (19)
\end{aligned}
$$

The above formulation contains a very large number of LGUBs (potentially exponential in $|B(t)|$ for all $t \in T$). To cope with this we proceed in a standard fashion by initially considering a subset of all inequalities and subsequently generating new inequalities when needed. In Section 4 we give the details of our column and row generation approach to solve the WND along with a heuristic routine for separating violated LGUBs inequalities (15). The overall behaviour of the row generation approach is strongly affected by the quality of the initial relaxation. In the context of WND, a particularly well-suited choice consists in including only the LGUBs (15) corresponding to interferer sets Γ with $|\Gamma| = 1$; we denote such initial relaxation by (PI^0). This choice has several major advantages.

First, the number of constraints in (PI^0) is small and can be generated efficiently. In the next section we actually show that, for each $t \in T, \beta \in B(t)$ and $\gamma \in B(t)\backslash\{\beta\}$, the number of non-dominated LGUBs (15) is at most $|L|$.

Second, as the Power-Indexed formulation (PI) is derived from the discretized SIR formulation (DM), so (PI^0) can be thought as derived from a relaxation (DM^0) of (DM). Namely, the relaxation (DM^0) is obtained from (DM) by replacing, for each $t \in T$ and each $\beta \in B(t)$, the SIR inequality (8) with the family of inequalities (one for each interferer):

$$
a_{t\gamma}\sum_{l\in L} P_l \cdot z_{\gamma l} - a_{t\beta}\sum_{l\in L} P_l \cdot z_{\beta l} + M \cdot x_{t\beta} \le \delta + M \qquad \gamma \in B(t)\backslash\{\beta\} \quad (20)
$$

Clearly, each inequality of type (20) is dominated by the original inequality (8) from which it derives, and the 0-1 solutions to (DM^0) may not be feasible for (DM). Nevertheless, in many applicative contexts (DM^0) appears to be a very good approximation of (DM). Indeed, this type of relaxation has been introduced in [19] to cope with DVB network design problems, and successfully applied to the design of the Italian national reference DVB network. Our experiments with WiMAX network design reported in Section 6 also confirms such good behaviour. Indeed, the number of inequalities not in PI^0 generated by our Branch-and-Cut is always very small. This can be well explained by the practical observation that in the downlink direction, for a given receiver there exists most of the times one particular interferer whose signal is much stronger than the others (see Sections 5 and 6 for a more detailed discussion).

A third and most crucial feature of PI^0 relates to the strength of its LGUBs inequalities. In next section we show that, for each $t \in T$, $\beta \in B(t)$ and $\gamma \in B(t)\backslash\{\beta\}$, the family of LGUBs associated with (20) along with the trivial facets define the corresponding *GUB knapsack polytope*, i.e. the convex hull of the 0-1 solutions to the knapsack SIR constraint (20) and its corresponding GUB constraints (9). This is a very desired property which explains why the LP-relaxations of PI^0 provide much tighter bounds than those provided by (DM^0), which in turn imply more effective searches and the capability to solve larger instances.

Summarizing, PI^0 can be easily generated, is a good approximation of the original problem and provides strong LP-relaxations.

3 The GUB knapsack polytope for the single-interferer SIR inequality

For a receiver $t \in T$, server $\beta \in B(t)$ and a single interferer $\gamma \in B(t)\backslash\{\beta\}$, let us consider the family of LGUBs associated with the constraint (20):

$$x_{t\beta} + \sum_{l=1}^{\lambda} z_{\beta l} + \sum_{j=q}^{|L|} z_{\gamma j} \leq 2 \qquad \lambda \in L,\ q \in L^I(t,\beta,\lambda,\{\gamma\}) \tag{21}$$

Since $P_l > P_{l-1}$ for $q = 2, \ldots, |L|$, the set $L^I(t,\beta,\lambda,\{\gamma\})$ of interfering levels of γ for a server power level λ can be written as $\{q(\lambda), q(\lambda)+1, \ldots, |L|\}$, where $q(\lambda) = \min\{l \in L : a_{t\gamma} \cdot P_l - a_{t\beta} \cdot P_\lambda > \delta\}$. It follows that the subfamily of inequalities (21) associated with λ is dominated by the single inequality corresponding to $q(\lambda)$. Finally, observe that $q(\lambda') \geq q(\lambda)$ for $\lambda' \geq \lambda$.

In order to simplify the notation, we now let $u = x_{t\beta}$, $v_l = z_{\beta l}$ for $l \in L$ and $w_l = z_{\gamma l}$ for $l \in L$. After removing the dominated LGUBs, the remaining family rewrites as:

$$u + \sum_{l=1}^{\lambda} v_l + \sum_{l=q(\lambda)}^{|L|} w_l \leq 2 \qquad \lambda = 1, \ldots, |L| \tag{22}$$

The following theorem extends a result presented in [29], also providing an alternative and simpler proof.

Theorem 2 *The polytope P defined as the set of points $(u, \mathbf{v}, \mathbf{w}) \in \mathbb{R}^{1+2|L|}$ satisfying (22) and the constraints $0 \leq u \leq 1$, $\mathbf{0} \leq \mathbf{v} \leq \mathbf{1}$ and $\mathbf{0} \leq \mathbf{w} \leq \mathbf{1}$ is the convex hull of the 0-1 solutions to (20).*

Proof. Proof of Theorem 2. Let A be the 0-1 coefficient matrix associated with the set of constraints (22). We first show that A is an *interval matrix*, i.e. in each column the 1's appear consecutively (see [23]).

We start by noticing that $A = (U|V|W)$ where U is the column associated with the variable u; $V \in \{0,1\}^{|L| \times |L|}$ is the square matrix associated with the variables $v_1, \ldots, v_{|L|}$; and $W \in \{0,1\}^{|L| \times |L|}$ is the square matrix associated with the variables $w_1, \ldots, w_{|L|}$.

The vector U has all the elements equal to 1 as u is included in every constraint (22). The matrix $V = [n_{ij}]$ with $i, j = 1, \ldots, |L|$ is lower triangular and such that $n_{ij} = 1$ for $i \geq j$. Indeed, the constraint (22) corresponding with $\lambda \in L$ includes exactly the v variables $v_1, \ldots, v_\lambda$.

Finally, consider the matrix $W = [m_{ij}]$ with $i, j = 1, \ldots, |L|$. First, observe that for all $\lambda, j \in L$, we have:

$$m_{\lambda j} = 1 \iff j \geq q(\lambda)$$

Recalling that for every $\lambda', \lambda \in L$ with $\lambda' \geq \lambda$, we have $q(\lambda') \geq q(\lambda)$, it follows that, for all $\lambda \leq \lambda'$, $m_{\lambda' j} = 1 \Longrightarrow j \geq q(\lambda') \Longrightarrow j \geq q(\lambda) \Longrightarrow m_{\lambda j} = 1$. W is thus an interval matrix and as U and V are interval matrices as well, it follows that A is an interval matrix and thus totally unimodular.

Finally, if we denote by B the matrix associated with the constraints (22) and the box constraints on variables $u, \mathbf{v}, \mathbf{w}$, then B is obtained by extending A with I and $-I$, where I is the identity matrix of size $1 + 2|L|$. Thus B is a totally unimodular matrix (see [23]) and, since the right hand sides of the constraints are integral, the vertices of P are also integral, completing the proof. $\square$

4 Solution Algorithm

The solution algorithm is based on the (PI) formulation to the WND and consists of two basic steps: (i) a set $\mathcal{P}$ of feasible power values is established; (ii) the associated formulation is solved by row generation and Branch-and-Cut. We start by describing step (ii) and we come back to step (i) later in this section.

In the following, for a fixed power set $\mathcal{P}$, we denote the solution algorithm for the associated (PI) formulation as SOLVE-PI($\mathcal{P}$). Since the (PI) formulation has in general an exponential number of constraints of type (15), we apply row generation. Namely, we start by considering only a suitable subset of constraints and we solve the associated relaxation. We then check if any of the neglected rows is violated by the current fractional solution. If so, we add the violated row to the formulation and solve again, otherwise we proceed with standard Branch-and-Cut (as implemented by the commercial solver ILOG Cplex [6]). The separation of violated constraints is repeated in each branching node.

At node 0, the initial formulation PI^0 includes only a subset of constraints (15), namely those including one interferer (i.e. $|\Gamma| = 1$). In Section 2 and Section 3 we discussed why this is a good choice for PI^0. Indeed, in our case studies, only a low number of additional constraints is added by separation during the iterations of the algorithm.

4.1 Separation.

We now proceed to show how violated constraints are separated. Let (x^*, z^*) be the current fractional solution. In Section 2 we have showed that constraints (15) are lifted GUB cover inequalities of (8). In order to separate a violated LGUB of type (15), we extend the standard heuristic approach to the separation of cover inequalities described in [23].

To this end, let us first select a receiver $t \in T$ and one of its servers, say $\beta \in B(t)$. We want to find a LGUB of type (15) that is associated with t and β, and is violated by the current solution (x^*, z^*). In other words, we want to identify a power level $\lambda \in L$ for β, a set of interferers $\Gamma = \{b_1, \ldots, b_{|\Gamma|}\} \subseteq B(t) \setminus \{\beta\}$ and an interfering $|\Gamma|$-tuple of power levels $q = (q_1, \ldots, q_{|\Gamma|}) \in L^I(t, \beta, \lambda, \Gamma)$, such that:

$$x^*_{t\beta} + \sum_{l=1}^{\lambda} z^*_{\beta l} + \sum_{i=1}^{|\Gamma|} \sum_{j=q_i}^{|L|} z^*_{b_i j} > |\Gamma| + 1 \tag{23}$$

Recall that $q \in L^I(t, \beta, \lambda, \Gamma)$ if

$$\sum_{i=1}^{|\Gamma|} a_{tb_i} \cdot P_{q_i} - a_{t\beta} \cdot P_\lambda > \delta \tag{24}$$

We solve the separation problem by defining a suitable 0-1 linear program. In particular, in order to identify a suitable pair (β, λ) we introduce, for every $l \in L$, a binary variable $u_{\beta l}$, which is 1 iff $l = \lambda$. Similarly, we introduce binary variables u_{bl} for all $b \in B(t) \setminus \{\beta\}$ and $l \in L$, with $u_{bl} = 1$ iff $(b, l) = (b_i, q_i)$, where $b_i \in \Gamma$ and q_i is the corresponding interfering power level. Then $u \in \{0, 1\}^{|B(t)| \times |L|}$ satisfies the following system of linear inequalities:

$$\sum_{b \in B(t) \setminus \{\beta\}} a_{tb} \sum_{l \in L} P_l \cdot u_{bl} - a_{t\beta} \sum_{l \in L} P_l \cdot u_{\beta l} > \delta \tag{25}$$

$$\sum_{l \in L} u_{bl} = 1 \qquad b \in B(t) \tag{26}$$

Constraint (25) ensures that u is the incidence vector of a cover of (8), whereas constraint (26) states that u satisfies the GUB constraints.

Observe now that $|\Gamma| = \sum_{b \in B(t) \setminus \{\beta\}} \sum_{l \in L} u_{bl}$. So, if u identifies a violated LGUB (23), we must have:

$$\sum_{l \in L} u_{\beta l} \sum_{k=1}^{l} z^*_{\beta k} + \sum_{b \in B(t) \setminus \{\beta\}} \sum_{l \in L} u_{bl} \sum_{k=l}^{|L|} z^*_{bk} \; > \; \sum_{b \in B(t) \setminus \{\beta\}} \sum_{l \in L} u_{bl} + 1 - x^*_{t\beta} \tag{27}$$

In order to (heuristically) search for a violated inequality, we proceed in a way which resembles the classical approach for standard cover inequalities (see [23]), by writing the following linear program (SEP):

$$Z \quad = \quad \max \quad \sum_{l \in L} u_{\beta l} \sum_{k=1}^{l} z^*_{\beta k} + \sum_{b \in B(t) \setminus \{\beta\}} \sum_{l \in L} u_{bl} \cdot \left(\sum_{k=l}^{|L|} z^*_{bk} - 1 \right) \qquad (SEP)$$

$$\text{s.t.} \quad \sum_{b \in B(t) \setminus \{\beta\}} a_{tb} \sum_{l \in L} P_l \cdot u_{bl} - a_{t\beta} \sum_{l \in L} P_l \cdot u_{\beta l} \geq \delta \qquad (28)$$

$$\sum_{l \in L} u_{bl} = 1 \qquad b \in B(t)$$

$$u_{bl} \geq 0 \qquad b \in B(t),\ l \in L$$

It is easy to notice that the feasible region of (SEP) contains all binary vectors satisfying (25) and (26). Let Z be the optimum value to (SEP). If $Z \leq 1 - x^*_{t\beta}$ then no binary vector u satisfies (27) and consequently no violated constraint exists. If $Z > 1 - x^*_{t\beta}$ then a violated constraint may exist, and we resort to a heuristic approach to find it. In particular, observe first that Z can be computed by relaxing the knapsack constraint (28) in a Lagrangian fashion and then by solving the resulting Lagrangian dual, namely:

$$Z \quad = \quad \min_{\eta \geq 0} \quad Z(\eta)$$

where $\eta \in \mathbb{R}^+$ is the Lagrangian multiplier and:

$$\begin{aligned} Z(\eta) \quad = \quad & \max_{u \geq 0} \quad \sum_{l \in L} u_{\beta l} \sum_{k=1}^{l} z^*_{\beta k} + \sum_{b \in B \setminus \{\beta\}} \sum_{l \in L} u_{bl} \cdot \left(\sum_{k=l}^{|L|} z^*_{bk} - 1 \right) \\ & + \quad \eta \cdot \left(\sum_{b \in B(t) \setminus \{\beta\}} a_{tb} \sum_{l \in L} P_l \cdot u_{bl} - a_{t\beta} \sum_{l \in L} P_l \cdot u_{\beta l} - \delta \right) \\ & \text{s.t.} \quad \sum_{l \in L} u_{bl} = 1 \qquad b \in B(t) \end{aligned}$$

For fixed $\eta \geq 0$, the objective $Z(\eta)$ can be easily computed by inspection. To simplify the notation we rewrite the objective function of the above linear program as:

$$-\delta \cdot \eta \quad + \quad \max_{u \geq 0} \sum_{b \in B(t)} \sum_{l \in L} c_{bl}(\eta) \cdot u_{bl} \qquad (29)$$

where, for every $b \in B(t), l \in L$, we let:

$$c_{bl}(\eta) = \begin{cases} \sum_{k=1}^{l} z^*_{\beta k} - \eta \cdot a_{t\beta} \cdot P_l & \text{if } b = \beta \\ \\ \sum_{k=l}^{|L|} z^*_{bk} - 1 + \eta \cdot a_{tb} \cdot P_l & \text{if } b \in B(t) \backslash \{\beta\} \end{cases}$$

For fixed $\eta \geq 0$, an optimal solution $u(\eta)$ to the inner maximization problem can be found by inspection as follows. For each $b \in B(t)$, identify a power level $l_b \in L$ which maximizes the coefficient in (29), namely $c_{bl_b}(\eta) = \max_{l \in L} c_{bl}(\eta)$; then, for each $b \in B(t)$ and each $l \in L$, let:

$$u_{bl}(\eta) = \begin{cases} 1 & \text{if } l = l_b \\ 0 & \text{otherwise} \end{cases}$$

It is straightforward to see that, for all $\eta \geq 0$, $u(\eta) \geq 0$ satisfies all constraints (26) and maximizes (29). For $\eta \geq 0$, the function $Z(\eta)$ is convex and unimodal (see [23]), and the optimum solution η^* can be found efficiently by applying the *Golden Section Search Method* (see [13]). Suppose now that $Z(\eta^*) > 1 - x^*_{t\beta}$ (otherwise no violated constraints exist). If, in addition, $u(\eta^*)$ also satisfies (25), then the positive components of the binary solution $u(\eta^*)$ are in one-to-one correspondence to the variables of a violated constraint. Otherwise the algorithm returns no violated cover.

4.2 The Algorithm

We come back now to the first step in our algorithm, namely the choice of the set of admissible power values $\mathcal{P}$. Large sets are in principle more likely to produce better quality solutions. However, the ability of the solution algorithm to find optimal or simply good-quality solutions is strongly affected by $|\mathcal{P}|$, as we will show in more detail in the computational section. Thus, the size and the elements of $\mathcal{P}$ should represent a suitable compromise between these two opposite behaviors. Moreover, the effectiveness of the Branch-and-Cut is typically affected by the availability of a good initial feasible solution. Thus, we decided to iteratively apply SOLVE-PI($\mathcal{P}$) to a sequence of power sets $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_r$. Each invocation inherits all the generated cuts, the best solution found so far and the corresponding lower bound from the previous invocation. More precisely, if we denote by -99 the switched-off state (in dBm), and $P^{dBm}_{\min}$, $P^{dBm}_{\max}$

are the (integer) minimum and maximum power values (in dBm), then we have $\mathcal{P}_0 = \{-99, P_{\max}^{dBm}\}$, $\mathcal{P}_1 = \{-99, P_{\min}^{dBm}, \left\lfloor \frac{P_{\max}^{dBm} - P_{\min}^{dBm}}{2} \right\rfloor, P_{\max}^{dBm}\}$ and $\mathcal{P}_r = \{-99, P_{\min}^{dBm}, P_{\min}^{dBm} + 1, \ldots, P_{\max}^{dBm}\}$. The structure of the intermediate power sets will be described in Section 6.

The overall approach, denominated *WPLAN*, is summarized in Algorithm 1, where i denotes the current iteration, along with the associated best solution found x_i, the corresponding value LB_i, and the set of feasible powers $\mathcal{P}_i$. If SOLVE-PI($\mathcal{P}_i$) is executed in less than the iteration time limit TL_i then the residual time τ_i is used to increase the time limit of the following iteration (i.e., TL_{i+1}:= TL_{i+1} + τ_i). The initial incumbent solution x_{-1} corresponds with all transmitters switched off and no receiver served ($LB_{-1} = 0$).

Algorithm 1 WPLAN

Input: the power sets $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_r$, the iteration time limit TL_i for $i = 0, \ldots, r$
Output: the best solution x_r
  $LB_{-1} := 0$
  for $i = 0$ to r **do**
    1. Invoke SOLVE-PI($\mathcal{P}_i$) with lower bound LB_{i-1}, incumbent x_{i-1} and TL_i
    2. Get x_i, LB_i and τ_i
    3. $TL_{i+1} := TL_{i+1} + \tau_i$
  end for
  Return x_r

5 WiMAX network design

The model introduced so far to solve the WND is a basic one and applies to most technologies, both in cellular and in broadcasting network design. Each technology is characterized by specific values for the constants appearing in the model and may require additional constraints and/or variables to model specific features as pointed out in Section 2.

In this section we introduce the technological elements and the modeling assumptions characterizing the specific technology addressed in this paper, namely the *IEEE Standard 802.16*, better known as *WiMAX* [28]. The major amendments concern the introduction of different frequency channels, channel capacity and traffic demand. In particular, each antenna emits at a specific frequency channel, and only co-channel signals are considered as interfering. Furthermore, a traffic demand is associated with each receiver, and the amount of total traffic served by a transmitter is limited by the channel capacity. We note that the resulting formulation incorporates the common features of the so-called *Next Generation Networks*, which adopt *Orthogonal Frequency Division Multiplexing* (*OFDM*) [27].

Specifically, we consider the design of a *Fixed WiMAX Network* [28] that provides broadband internet access. The network consists of a set of installations - the *base stations* (*BS*) - distributed over a number of *sites* in order to provide connectivity to a set of customers' equipment - the *subscriber stations* (*SS*) - located in a target area. The network is based on *frequency division duplexing (FDD)*, and thus transmissions from BSs to SSs (*downlink*) and transmissions from SSs to BSs (*uplink*) take place on two separate frequency bands [3].

The target area is decomposed into a grid of approximately squared elementary areas called *testpoints (TP)*. All SSs located in a TP are aggregated in a single fictitious SS located in the centre of the TP. Each TP thus corresponds to a single receiver and the set of all the TPs corresponds to the set of receivers T in the basic model. For each TP $t \in T$ we introduce the quantity d_t to represent the joint bandwidth request (traffic demand) of all the SSs located in t.

A BS typically consists of a pylon accommodating a number of *transceivers (TRX)*. The set of all the TRXs that can be deployed in the target area corresponds to the set of transmitters B of our basic model. Every TRX $b \in B$ is characterized by a position (the TP in which the TRX is located) and by two radio-electrical parameters: *i*) *frequency channel* f, which belongs to a finite set of available channels F, each having a constant bandwidth D; *ii*) *emitted power* $P_b^f \in [P_{\min}, P_{\max}]$ on frequency $f \in F$.

Just like other Next Generation Networks, WiMAX supports the so-called *Adaptive Modulation and Coding* (AMC), which allows transmission scheme (*burst profile*) to change according to radio channel condition [3]. Each TRX can select a specific burst profile to serve each TP. The selected burst profile affects both the SIR threshold and the fraction of channel capacity exploited to fulfill the traffic demand of a TP. So, by denoting the set of available burst profiles as H, we introduce two new parameters for every

$h \in H$: the *SIR threshold* δ'_h that must be satisfied to ensure service coverage according to (2), and the *spectral efficiency* s_h, which is the bandwidth required to satisfy one unit of demand.

We are now able to write a modified version of the SIR inequality that takes into account the WiMAX specific features. In particular, TP $t \in T$ is served by TRX $\beta \in B(t)$ if the following constraint is satisfied:

$$\sum_{b \in B(t) \setminus \{\beta\}} a_{tb} \cdot p_b^{f(\beta)} - a_{t\beta} \cdot p_\beta^{f(\beta)} \leq \delta_{h(t)}. \tag{30}$$

where $f(\beta) \in F$ is the transmission frequency assigned to β, whereas $h(t) \in H$ is the burst profile used to serve t.

We remark that the SIR inequality (30) models the coverage condition for the *downlink* direction. As a consequence, it only involves the power variables of the BSs. Downlink is in general most critical in applications such as internet services [2, 27] and is the unique direction in broadcasting networks [20]. Our focus on downlink is motivated by the fact that we are considering the design of a fixed WiMAX network that provide broadband internet access. The uplink direction is more critical in mobile networks and can still be modeled by the SIR inequality (30) with power variables corresponding to the emissions of the SSs. The results that we presented in Section 2 can therefore be applied also to model the uplink.

If we denote by $T(\beta)$ the family of testpoints served by $\beta \in B$, the limited channel capacity is expressed by the following constraint:

$$\sum_{t \in T(\beta)} d_t \cdot \frac{1}{s_h} \leq D \tag{31}$$

In order to represent these new features into our basic 0-1 program, we need to introduce new binary variables, obtained by slightly modifying the original ones to take into account multiple frequencies and burst profiles. We thus let:

$$x_{tb}^{fh} = \begin{cases} 1 & \text{if testpoint } t \in T \text{ is served by TRX } b \in B \\ & \text{on frequency } f \in F \text{ with burst profile } h \in H \\ 0 & \text{otherwise} \end{cases}$$

$$z_{bl}^{f} = \begin{cases} 1 & \text{if TRX } b \in B \text{ emits at power level } l \in L \text{ on frequency } f \in F \\ 0 & \text{otherwise} \end{cases}$$

We also need to introduce a new version of the set of interfering levels $L^I(t,\beta,\lambda,\Gamma)$, that now depends also on the used burst profile $h \in H$, in addition to the TP $t \in T$, the server $\beta \in B(t)$, the server emitted power $\lambda \in L$ and the set of interferers $\Gamma \subseteq B(t)\backslash\{\beta\}$:

$$L^I(t,\beta,h,\lambda,\Gamma) = \{q \in L^{|\Gamma|} : \sum_{i=1}^{|\Gamma|} a_{tb_i} \cdot P_{q_i} - a_{t\beta} \cdot P_\lambda > \delta_h\}$$

We can finally state the Power-Indexed formulation for WiMAX network design:

$$\max \quad \sum_{t\in T} \sum_{b\in B(t)} \sum_{f\in F} \sum_{h\in H} r_t \cdot x_{tb}^{fh} \qquad (WiMAX-PI)$$

$$\text{s.t.} \quad x_{tb}^{fh} + \sum_{l=1}^{\lambda} z_{\beta l}^{f} + \sum_{i=1}^{|\Gamma|} \sum_{j=q_i}^{|L|} z_{b_i j}^{f} \le |\Gamma| + 1 \qquad t \in T,\ \beta \in B(t),\ f \in F,\ h \in H,\ \lambda \in L,\ \Gamma \subseteq B(t)\backslash\{\beta\},\ q \in L^I(t,\beta,h,\lambda,\Gamma) \qquad (32)$$

$$\sum_{b\in B(t)} \sum_{f\in F} \sum_{h\in H} x_{tb}^{fh} \le 1 \qquad t \in T \qquad (33)$$

$$\sum_{l\in L} z_{bl}^{f} = 1 \qquad b \in B,\ f \in F \qquad (34)$$

$$\sum_{t\in T} \sum_{h\in H} d_t \cdot \frac{1}{s_h} \cdot x_{tb}^{fh} \le D \qquad b \in B,\ f \in F \qquad (35)$$

$$x_{tb}^{fh} \in \{0,1\} \qquad t \in T,\ b \in B(t),\ f \in F,\ h \in H \qquad (36)$$

$$z_{bl}^{f} \in \{0,1\} \qquad b \in B,\ l \in L,\ f \in F \qquad (37)$$

Note that constraint (35) models the capacity constraint (31). All other constraints are simple generalizations of the basic ones.

6 Computational Results

In this section we present computational results over a set of realistic instances, developed with the Technical Strategy & Innovations Unit of British Telecom (BT).

The target of these tests is manyfold. First, we compare the new (PI) formulation to the two big-M formulations (BM) and (DM) and show that (PI) outperforms (BM) and (DM) in terms of quality of produced bounds and solutions found. Then, we illustrate specific features of the solution algorithm WPLAN and we motivate the iterative approach with increasing power sets. Finally, we assess the ability of WPLAN to tackle realistic WiMAX network design instances. The tests were performed under Windows XP 5.1 operating system, with 1.80 GHz Intel Core 2 Duo processor and 2×1024 MB DDR2-SD RAM. The algorithm is implemented in C++ (under Microsoft Visual Studio 2005 8.0), whereas the commercial MILP solver ILOG Cplex 10.1 is invoked by ILOG Concert Technology 2.3.

6.1 The test-bed

All our instances correspond to an urban area located in the North Eastern part of Rome (Italy) selected in agreement with the engineers at BT, who considered it as a representative residential traffic scenario. All instances are available online [9].

Physical data of the target area are provided by a *Digital Elevation Model (DEM)* that represents the territory as a raster with a resolution of about 100 meters. The set of instances refers to an area of about 2.5 Km $\times$ 2.5 Km, corresponding to a residential neighborhood of Rome: according to the DEM resolution, the area is decomposed into a 25×25 testpoints grid. Fifteen instances are drawn out of this basic area, that is classified as an urban environment.

In conformity with the regulations established by the *Italian Communications Regulatory Authority (Agcom)* for the deployment of WiMAX networks in Italy [1], we carry out the planning study for one of the provided transmission licenses. The frequency set F thus includes three 7 MHz channels in the (3.4 to 3.6) GHz band.

A set H of four burst profiles is available for transmissions and the bandwidth demand d_t of each testpoint $t \in T$ is estimated according to the methodology described in [31], considering an urban scenario where customers are mainly residential.

On the basis of the target area size and considering an average spacing of about 0.5 Km, a number of potential BSs can be activated: each BS may install up to 3 directional TRXs with 120° antennas emitting in the power range [20,40] dBm. We refer to commercial devices operating in the 3.5 GHz frequency band. The azimuth of each antenna may vary in the range $[0°, 360°]$ with a step of 10°, thus allowing 36 distinct orientations for each TRX. So, in principle, we may have up to 12 different orientations associated with each directional TRX. However, as in [11], in order to limit the size of the instances, we choose to reduce the number of possible installations by selecting one most promising orientation in advance: for each directive antenna, we select the direction which maximizes coverage (an exact description of the selection strategy can be found at our WiMAX web page [9]).

The fading coefficients $\tilde{a}_{tb}$ are computed by means of the path loss model *COST-231 Hata* [5], that is widely used and taken as reference for predictions in WiMAX networks [30]. However, we note that the optimization model is independent of the particular propagation model that is used, as it only affects the coefficients of the fading matrix.

We define three types of instances, denoted by Sx with x $= \{1, \ldots, 7\}$, Rx with x $= \{1, \ldots, 4\}$ and Qx with x $= \{1, \ldots, 4\}$. For the Sx instances, the traffic is uniformly distributed among the TPs and we assign unitary revenue to each TP (i.e. $r_t = 1$). Finding an optimal coverage plan thus corresponds to define the plan with the maximum number of covered TPs. Only one frequency and one burst profile are allowed. For the Rx instances, we consider a traffic distribution based on the actual distribution of the buildings. We also introduce multiple frequencies and burst profiles. In this case, the revenue of each testpoint is proportional to the traffic generated. Finally, the Qx instances include an increasing number of candidate sites and focus on a single frequency network with multiple burst profiles.

6.2 Numerical Results and Comparisons

We have pointed out in Section 1 that the solutions to (BM) and (DM) returned by state-of-the-art MILP solvers such as Cplex can be affected by numerical inaccuracy, i.e. the SIR inequalities of testpoints recognized

Table 1: Description of the test-bed instances

ID	|T|	|B|	|F|	|H|
S1	100	12	1	1
S2	169	12	1	1
S3	196	12	1	1
S4	225	12	1	1
S5	289	12	1	1
S6	361	12	1	1
S7	400	18	1	1
R1	400	18	3	4
R2	441	18	3	4
R3	484	27	3	4
R4	529	27	3	4
Q1	400	36	1	4
Q2	441	36	1	4
Q3	484	36	1	4
Q4	529	36	1	4

as covered are actually unsatisfied (similar problems were also reported in [16] and [19]). We detect such coverage errors by evaluating the solutions *off-line*: after the optimization process, we verify that the SIR inequality corresponding to each nominally covered testpoint is really satisfied by the power vector of the returned solution. This is not the only issue, as in the case of (DM) the problem can be even wrongly evaluated as infeasible.

We experienced that tuning the parameters of Cplex is crucial to reduce coverage errors and to contain the effects of numerical instability. Furthermore, in the case of (DM), tuning is essential to ensure that the problem is correctly recognized as feasible. After a series of tests, we established that an effective setting consists of turning off the *presolve* and on the *numerical emphasis*. Moreover, we turn off the generation of the *mixed-integer rounding cuts* and of the *Gomory fractional cuts*.

Assessing the strength of the Power-Indexed formulation.

The first group of experiments aims to assess the higher strength of (PI) and compare it with the strength of (BM) and (DM). To this end, we focus on a single instance of our test-bed (instance S4 presented in Table 1) and detail the behaviour of WPLAN for each invocation of SOLVE-PI($\mathcal{P}$). The sets of power values in the first three invocations of SOLVE-PI($\mathcal{P}$) are (in dBm)

$\mathcal{P}_1 = \{-99, 40\}, \mathcal{P}_2 = \{-99, 20, 30, 40\}$ and $\mathcal{P}_3 = \{-99, 20, 25, 30, 35, 40\}$, respectively. Then, in each of the following invocations, $\mathcal{P}$ is expanded by including two more values (suitably spaced). To analyse the behaviour of the single iterations and establish an effective sequence of power sets, we set a time limit of 1 hour for each invocation of the solution algorithm for (PI) and (DM).

In order to evaluate the quality of (PI) w.r.t. (DM), we apply WPLAN to (DM) (note that in this case the solution procedure SOLVE-PI is replaced by the simple solution of the (DM) by Cplex). In Table 2, for each iteration of WPLAN, we report the number |L| of considered power levels, the number of LGUBs included in the initial formulation (PI^0) and the number of LGUBs separated during the current iteration. Additionally, for both (PI) and (DM), we report the upper bound at node 0 (UB), the value |T*| of the final solution (number of covered testpoints) and the final gap. When the solution contains coverage errors, two values are presented in the |T*| column, namely the nominal value of the best solution returned by Cplex (in brackets) and its actual value computed by re-evaluating the solution *off-line*.

The last line of the table shows the results obtained for (BM) by setting a time limit of 3 hours. Note that in this case, the second column reports the number of SIR (big-M) constraints (5) included in (BM).

The figures in Table 2 are representative of the typical behaviour of WPLAN on all instances of our test-bed. They allow us to make some relevant observations. First, the size of (PI) grows quickly with the number of power levels, and is typically much larger than that of (BM) and (DM). This is counterbalanced by the quality of the upper bounds, which are consistently better for (PI) and, most important, the quality of the solutions found. Interestingly, the best solution is found quite early in the iterative procedure, namely for $|\mathcal{P}| \leq 6$. A similar behaviour is observed for the other instances reported in Table 3 as well. This motivated our choice of the sequence of feasible power values in the final version of WPLAN: most of the computational effort is concentrated on small cardinality power sets, and only one large set. More precisely, there will be only 4 iterations, corresponding to 2, 4, 6 and 22 power levels, respectively.

Finally, we note that the number of generated LGUBs is small. Also, in most cases the LGUBs include only two interferers, and in any case never

Table 2: Behaviour of WPLAN for instance S4

|L|	LGUBs init	LGUBs added	(PI) UB	(PI) |T*|	(PI) gap%	(DM) UB	(DM) |T*|	(DM) gap%
2	5743	17	199.2193	106	0.00	218.3465	91	125.65
4	9035	7	204.2500	111	0.00	219.0015	97 (98)	102.68
6	14312	13	206.6261	111	59.03	219.3488	100 (101)	115.70
8	17142	45	209.4200	111	67.51	219.7349	100 (101)	122.98
10	24638	6	210.0000	111	79.99	220.2788	100 (101)	123.14
12	27799	1	211.7000	111	82.05	219.9144	100 (101)	124.01
14	35944	0	212.0000	111	83.46	220.1307	100 (101)	123.58
16	38496	10	214.5930	111	85.48	220.3000	100 (101)	125.00
18	45425	2	215.8000	111	86.44	220.1091	100 (101)	124.83
20	48918	2	218.0000	111	89.99	220.0560	100 (101)	125.00
22	57753	3	218.0000	111	90.83	220.3720	100 (101)	125.00
(BM)	1170	-	221.3925	93	97.18	-	-	-

more than three. In other words, even though many interferers can reach a given testpoint, only very few of them (in most cases only one) give a significant contribution to the overall interference. As already pointed out, we remark that such behaviour is typical of the downlink direction, where it is very common to have in a given testpoint a (very) small number of interfering signals which are significantly stronger than the others.

The performance of the Power-Indexed approach over the test-bed.

After having shown the higher strength of (PI), we now proceed to report and comment the full set of results over our benchmark instances (Table 3). In this case, we set a time limit of 3 hours for the solution of both (BM) and (DM) and for WPLAN applied to (PI). For (DM) we use the full set of power levels. The value of the best solutions found within the time limit are shown in column |T*|. The *gap* columns report the gap between the upper and lower bound at termination, whereas the last column |L*| is the number of power levels used in the iteration in which WPLAN obtains the best solution.

The results show that WPLAN applied to (PI) outperforms (BM) and (DM) in terms of quality of the solutions found and running times to obtain

Table 3: Comparisons between (BM) and WPLAN formulations

ID	|T|	(BM)			(DM)			WPLAN		
		|T*|	gap%	time (sec)	|T*|	gap%	time (sec)	|T*|	time (sec)	|L*|
S1	100	63 (78)	43.72	10698	60 (62)	61.29	10776	74	10565	6
S2	169	99 (100)	56.18	10705	58 (63)	191.38	10791	107	5591	4
S3	196	108	79.54	4010	49 (52)	300.00	201	113	5732	4
S4	225	93	103.43	10761	90	147.78	7424	111	7935	4
S5	289	77	202.24	10002	70 (81)	312.86	2860	86	10329	6
S6	361	154	130.76	8110	125 (244)	188.80	7535	170	8723	4
S7	400	259 (266)	53.67	8860	91 (94)	339.56	1765	341	7154	4
R1	400	370	7.57	10626	Out	-	-	400	1579	2
R2	441	302 (303)	45.03	3595	Out	-	-	441	1244	4
R3	484	99 (99)	385.86	10757	Out	-	-	427	3472	2
R4	529	283 (286)	84.96	10765	Out	-	-	529	2984	2
Q1	400	0	-	-	Out	-	-	67	2756	2
Q2	441	191	9124	130.89	Out	-	-	211	7132	4
Q3	484	226	112.83	3392	Out	-	-	463	3323	2
Q4	529	145 (147)	264.83	6623	Out	-	-	491	3053	2

them. Even if in principle the reduced and quite small number of power values considered by WPLAN could result in poorer coverage w.r.t. (BM), the figures clearly show that this is not the case. On one hand, this happens as a small number of well-spaced power values suffices in practice to obtain good coverage; indeed, it is common practice in WiMAX network planning to neglect intermediate values, i.e. a device is either switched-off or activated at its maximum power [26]. On the other hand, the size of the (BM) formulation and the ill-conditioned constraint matrix, along with the presence of the big-M coefficients, makes the solution process quite unstable, the solutions found unreliable and the branching tree extremely large. Indeed, due to rounding errors and numerical instability, several solutions to (BM) turn out to be infeasible when verified off-line.

WPLAN applied to (PI) also outperforms (DM) with all the power levels included. The results clearly show that the simple discretization of the power range does not suffice to get better solutions than those obtained by (BM). Indeed, when all power levels are considered, the performance of (DM) is even worse than that of (BM) and coverage errors are still present. This is particularly evident in the case of instance S3: the solution of (DM) ensures less than half the coverage of the solution of (BM). We also note that the number of wrongly evaluated coverage conditions may be high (instance S6). In the case of the instances with multiple frequencies and burst profiles, the resulting MILP instance of (DM) is so large that Cplex runs out of memory while building it (clearly this happens as well with (PI) when 22 power levels are considered). It is furthermore interesting to note that the improvement of the upper bound within the time limit is small (*tailing off*) and the gap is reduced just by improving the incumbent feasible solution.

All the difficulties that we pointed out are overcome by the Power-Indexed formulation (PI) and the solution approach WPLAN. Coverage errors, in particular, are completely eliminated. The higher performance of our approach is especially apparent for the R-instances, which seem to be quite easy for WPLAN but very difficult for (BM) and (DM). Indeed, when no time limit is imposed to the solution of (BM), Cplex runs out of memory after about ten hours of computation without getting sensible improvements in the bounds. On the contrary, in the case of R1, R2 and R4 SOLVE-PI($\mathcal{P}$) finds the optimum solution (when $|\mathcal{P}| = 2$) in less than 1 hour. The higher performance is also highlighted in the case of instance Q1 that turns out to be hard: both (BM) and (DM) with 2 power levels cannot find any feasible solution with non-zero value within the time limit, while, in contrast, (PI) founds a solution with value 67.

Comparisons between warm and cold start for *(PI)*.

Finally, in Table 4 we show the impact of the iterative approach WPLAN on the quality of the solutions found for (PI). In particular we compare *cold starts*, which correspond to invoking SOLVE-PI($\mathcal{P}$) without benefitting from cuts and lower bounds obtained at former invocations, with *warm starts* which, in contrast, make use of such information. The value of the best solutions found during successive invocations of SOLVE-PI both under warm and cold starts are shown in the columns identified by $|L| = n$, where n denotes the number of corresponding power levels. The value of the best solution found at the first invocation is in column $|L| = 2$, while the value of the best solution and the number of levels used to find it are shown in column $|T^*|$ and $|L^*|$, respectively.

For all S-instances the best solution can be found only thanks to warm start. Note that SOLVE-PI encounters increasing difficulties in finding good

Table 4: Comparisons between warm and cold starts

ID	|T*|	|L*|	|L|=2	WARM START		COLD START	
				|L|=4	|L|=6	|L|=4	|L|=6
S1	74	6	69	72	74	71	58
S2	107	4	72	107	107	80	63
S3	113	4	83	113	113	108	101
S4	111	4	75	111	111	100	97
S5	86	6	76	84	86	83	81
S6	170	4	127	170	170	110	127
S7	341	4	296	341	341	314	196
R1	400	2	400	-	-	399	304
R2	441	4	416	441	-	394	355
R3	427	2	427	427	427	414	Out
R4	529	2	529	-	-	512	Out
Q1	67	2	67	67	67	$*$	$*$
Q2	211	4	196	211	211	156	Out
Q3	463	2	463	463	463	Out	Out
Q4	491	2	491	491	491	Out	Out

solutions as the number of power levels increases (in the case of the apparently hard instance Q1, for 3 and 5 power levels, no feasible solution is found within the time limit when cold start is adopted). This is mainly due to the large size of the corresponding instances, that, in some cases denoted by *Out*, makes Cplex run out of memory while building the model. However, a good initial solution provided to SOLVE-PI can be improved in most cases. We have already observed that for a larger number of levels (i.e. > 6), no improved solutions can be found for all instances in our test-bed. Finally, for R1 and R4 a solution covering the entire target area is found already with $|L| = 2$, while for R2 such a solution is found with $|L| = 4$ (and warm-start).

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References

[1] AGCOM, Italian Communications Regulatory Authority. 2007. *Procedure per l'assegnazione di diritti d'uso di frequenze per sistemi Broadband Wireless Access (BWA) nella banda a 3.5 GHz - Delibera n. 209/07/CONS.* http://www2.agcom.it/provv/d_209_07_CONS/d_209_07_CONS.htm.

[2] Amaldi, E., P. Belotti, A. Capone, F. Malucelli. 2006. Optimizing base station location and configuration in UMTS networks. *Ann. Oper. Res.* **146** (1) 135–152.

[3] Andrews, J.G., A. Ghosh, R. Muhamed. 2007. *Fundamentals of WiMAX.* Prentice Hall, Upper Saddle River, USA.

[4] Castorini, E., P. Nobili, C. Triki. 2008. Optimal Routing and Resource Allocation in Ad-Hoc Networks. *Opt. Meth. Soft.* **23** (4) 593–608.

[5] COST Action 231 - Digital Mobile Radio Towards Future Generation Systems. 1999. *COST 231 Final Report.* http://www.lx.it.pt/cost231/final_report.htm.

[6] ILOG Cplex. http://www.ilog.fr.

[7] D'Andreagiovanni, F. 2010. Pure 0-1 Programming Approaches to Wireless Network Design. *Ph.D. Thesis, Sapienza - Università di Roma, Roma, Italy.*

[8] D'Andreagiovanni, F., C Mannino. 2009. An Optimization Model for WiMAX Network Planning. *WiMAX Network Planning and Optimization*, Ed. Y. Zhang, Auerbach Publications, Boca Raton, USA, 369–386.

[9] http://www.dis.uniroma1.it/~fdag/WNP, Fabio D'Andreagiovanni's Academic Website.

[10] Dyer, M., L. Wolsey. 1990. Formulating the single machine sequencing problem with release dates as a mixed integer program. *Discrete Appl. Math.* **26** (2-3) 255–270.

[11] Eisenblätter, A., H. Geerdes, T. Koch, A. Martin, R. Wessäly. 2006. UMTS Radio Network Evaluation and Optimization Beyond Snapshots. *Math. Meth. Oper. Res.* **63** (1) 1–29.

[12] Fridman, A., S. Weber, K.R. Dandekar, M. Kam. 2008. Cross-Layer Multicommodity Capacity Expansion on Ad Hoc Wireless Networks of Cognitive Radios. *of the 42nd Conference on Information Sciences and Systems (CISS)*, Princeton, USA, 676–680.

[13] Gerald, C.F., P.O. Wheatley. 2004. *Applied Numerical Analysis, 7th Edition.* Addison-Wesley, Upper Saddle River, USA.

[14] Hata, M. 1980. Empirical Formula for Propagaton Loss in Land Mobile Radio Services. *IEEE Trans. Vehic. Tech.* **29** 317–325.

[15] Johnson, E.L., M.W. Padberg. 1981. A note on the knapsack problem with special ordered sets. *Oper. Res. Letters* **1** (1) 18–22.

[16] Kalvenes, J., J. Kennington, E. Olinick. 2006. Base Station Location and Service Assignments in W-CDMA Networks. *INFORMS J. Comp.* **18** (3) 366–376.

[17] Mallinson, M., P. Drane, D. Hussain. 2007. Discrete radio power level consumption model in wireless sensor networks. *Proceedings the Second International Workshop on Information Fusion and Dissemination in Wireless Sensor Networks (Sensor Fusion)*, Pisa, Italy.

[18] Mannino, C., F. Marinelli, F. Rossi, S. Smriglio. 2007. A Tight Reformulation of the Power-and-Frequency Assignment Problem in Wireless Networks. *Technical Report TRCS 003/2007*, Dipartimento di Informatica, Università dell'Aquila, L'Aquila, Italy.

[19] Mannino, C., S. Mattia, A. Sassano. 2009. Wireless Network Design by Shortest Path. *Comp. Opt. Appl.*, to appear.

[20] Mannino, C., F. Rossi, S. Smriglio. 2006. The Network Packing Problem in Terrestrial Broadcasting. *Oper. Res.* **54** (6) 611–626.

[21] Mathar, R., M. Schmeinck. 2005. Optimisation Models for GSM Radio. *Int. J. Mob. Net. Des. and Innov.* **1** (1) 70–75.

[22] Naoum-Sawaya J. 2007. New Benders' Decomposition Approaches for W-CDMA Telecommunication Network Design. *Ph.D. Thesis, University of Waterloo, Waterloo, Canada.* http://www.uwspace.uwaterloo.ca/handle/10012/2769.

[23] Nehmauser G., L. Wolsey. 1988. *Integer and Combinatorial Optimization.* John Wiley & Sons, Hoboken, USA.

[24] Parrello, E. 2006. Models and algorithms for wireless network planning. *Ph.D. Thesis, Sapienza - Università di Roma, Rome, Italy.*

[25] Rappaport, T.S. 2001. *Wireless Communications: Principles and Practice, 2nd Edition.* Prentice Hall, Upper Saddle River, USA.

[26] Ridolfi, S., Senior Network Engineer - British Telecom Italia (BT). 2010. Personal communication.

[27] Shao L., S. Roy. 2005. Downlink Multicell MIMO-OFDM: an Architecture for Next Generation Wireless Networks. *IEEE Wireless Communications and Networking Conference - WCNC2005*, New Orleans, USA, **2** 1120–1125.

[28] IEEE Std. 802.16-2004, IEEE Standard for Local and Metropolitan Area Networks, "Part 16: Air Interface for Fixed Broadband Wireless Access System", 2004.

[29] Wolsey, L, 1990. Valid inequalities for 0-1 knapsacks and mips with generalised upper bound constraints. *Discrete Appl. Math.* **29** (2-3) 251–261.

[30] IEEE 802.16 Broadband Wireless Access Working Group, "Channel Models for Fixed Wireless Apllications System", 2003.

[31] "WiMAX Deployment Considerations for Fixed Wireless Access in the 2.5 GHz and 3.5 GHz Licensed Bands", WiMAX Forum, White Papers, June 2005.